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CONCENTRATIONAL POLARIZATION IN ULTRAFILTRATION IN A PLANE CHANNEL

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We investigated steady-state concentrational polarization in laminar ultrafiltration in a plane channel in relation to the selective properties of the membrane.

In the existing literature there is a lack of any clear theoretical ideas about laminar continuous-flow ultrafiltration in plane channels with imperfect selectivity of the membranes due to the complexity of the differential equations describing the process. There are individual works available [1−3] in which, based on the integral method, attempts are made to analyze the process of ultrafiltration for the particular case of ideal selectivity of the membrane. The integral method used for describing ultrafiltration suffers from of important limitations and deficiencies consisting in the impossibility of using power-law or polynomial distributions of the concentration field to simultaneously describe accurately both the concentration profile and the diffusion flow, which is required by virtue of the specific features of the boundary condition on the membrane.

On the basis of a semiintegral method suggested in [4], which is devoid of the aforementioned drawback, in the present work we investigated the process of concentrational polarization in laminar ultrafiltration for the most general case of nonideal selectivity of the membrane. Here, we confine our considerations to the first stage of ultrafiltration, in which gel is not formed on the surface of the membrane. Then concentrational polarization in laminar continuous-flow ultrafiltration in a plane slit channel is described by the dimensionless equation of convective diffusion

$$u\frac{\partial\Theta}{\partial\xi}+v\frac{\partial\Theta}{\partial\eta}=\frac{1}{\mathrm{Pe}}\frac{\partial^2\Theta}{\partial\eta^2} \tag{1}$$

with the boundary conditions

$$\varphi V\Theta+\frac{1}{\mathrm{Pe}}\frac{\partial\Theta}{\partial\eta}=0\ \ (\eta=0)\,;\quad \frac{\partial\Theta}{\partial\eta}=0\ \ (\eta=1)\,; \tag{2}$$

$$\Theta=1\ \ (\xi=0)\,. \tag{3}$$

In [5] it is shown that in the case of small transmembrane velocities V (which are precisely realized in ultrafiltration) and parabolic velocity distribution at the inlet to a plane channel it is possible to use, with a high degree of accuracy, the following expressions for the velocity components u and v:

$$u=3\,(1-V\xi)\left(\eta-\frac{\eta^2}{2}\right),\quad v=-V\left(1-\frac{3}{2}\eta^2+\frac{\eta^3}{2}\right). \tag{4}$$

If we take into account the fact that in ultrafiltration Pe >> 1 ($\mathrm{Pe}\sim10^7$), then the change in the concentration occurs entirely within the limits of a diffusional boundary layer lying in a narrow near-membrane

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region. Then, to solve the equation of convective diffusion (1) we can restrict ourselves to the first terms in η in the velocity distributions (4):

$$3(1-V\xi)\eta\frac{\partial\Theta}{\partial\xi}=\frac{\partial}{\partial\eta}\left(V\Theta+\frac{1}{\mathrm{Pe}}\frac{\partial\Theta}{\partial\eta}\right). \tag{5}$$

In the immediate vicinity of the membrane, where $\eta\to 0$, the concentration distribution, which in many respects determines the process of concentrational polarization, can be found from the equation

$$\frac{\partial}{\partial\eta}\left(V\Theta+\frac{1}{\mathrm{Pe}}\frac{\partial\Theta}{\partial\eta}\right)\approx 0, \tag{6}$$

which follows directly from relation (5). After double integration of Eq. (6) and satisfaction of the first boundary condition of (2) we find

$$\Theta=\Theta_w\left[1-\varphi+\varphi\exp(-\mathrm{Pe}\,V\eta)\right], \tag{7}$$

where $\Theta_w(\xi)$ is the presently unknown concentration of the dissolved substance on the membrane surface. To find it, we shall avail ourselves of the integral equation of mass balance. For this, we integrate the equation of convective diffusion (5) across the boundary layer from 0 to δ and take into account the first boundary condition of (2) together with the requirement that $\Theta = 1$ at $\eta = \delta$. This will yield

$$\frac{d}{d\xi}\int_0^\delta 3(1-V\xi)\eta(\Theta-1)\,d\eta+(1-\varphi)V\Theta_w=V. \tag{8}$$

Starting with Eq. (7) and physical considerations based on the concept of the boundary layer, we will represent the distribution of the concentration Θ by the relation

$$\Theta=\begin{cases}\Theta_w(\xi)\left[1-\varphi+\varphi\exp(-\mathrm{Pe}\,V\eta)\right], & 0\le\eta\le\delta(\xi),\\ 1, & \delta(\xi)<\eta\le 1,\end{cases} \tag{9}$$

where the thickness of the diffusion boundary layer $\delta(\xi)$ will be determined from the condition

$$\Theta_w\left[1-\varphi+\varphi\exp(-\mathrm{Pe}\,V\delta)\right]=1.$$

From this we obtain

$$\delta=\frac{1}{\mathrm{Pe}\,V}\ln\frac{\varphi\Theta_w}{1-(1-\varphi)\Theta_w}. \tag{10}$$

We substitute Eqs. (9) and (10) into the relation (8) of mass balance. This will give an equation for computing concentration $\Theta_w(\xi)$ on the membrane surface:

$$\frac{d}{dV\xi}\left\{(1-V\xi)\left(\Theta_w-1-\left[1-(1-\varphi)\Theta_w\right]\ln\frac{\varphi\Theta_w}{1-(1-\varphi)\Theta_w}-\right.\right.$$

$$\left.\left.-\frac{1}{2}\left[1-(1-\varphi)\Theta_w\right]\left(\ln\frac{\varphi\Theta_w}{1-(1-\varphi)\Theta_w}\right)^2\right)\right\}=\frac{\mathrm{Pe}^2V^2}{3}\left[1-(1-\varphi)\Theta_w\right]. \tag{11}$$

In the general case, one cannot integrate Eq. (11). Let us consider some very important specific cases.

At small distances from the entrance to the channel, where $V\xi \ll 1$ and $\Theta_w \sim 1$, the latter quantity can be represented in the form $\Theta_w = 1 + \varepsilon$, where $\varepsilon < 1$. Then $\ln \varphi\Theta_w = \ln \varphi(1+\varepsilon)$, $\ln\,[1-(1-\varphi)\Theta_w] = \ln \varphi\,[1-\varepsilon(1-\varphi)/\varphi]$, and, expanding these logarithms in a series in powers of ε, we obtain

$$\frac{d\varepsilon^3}{dV\xi} = 2\varphi^3\,\mathrm{Pe}^2\,V^2\left[1-\frac{\varepsilon\,(1-\varphi)}{\varphi}\right].$$

From this, assuming that $\varepsilon(1-\varphi)/\varphi \ll 1$, we find

$$\Theta_w = 1 + \varphi\,(2\,\mathrm{Pe}^2V^2V\xi)^{1/3}. \tag{12}$$

Determining the thickness of the diffusional boundary layer in the form of Eq. (10), when $1-(1-\varphi)\Theta_w \to 0$, we obtain $\delta = (1/\mathrm{Pe})\ln\,[\varphi\Theta_w/(1-\varphi)\Theta_w)] \to \infty$, and one might think that Eq. (11) cannot be used for computing Θ_w. However, the factor preceding the logarithm tends to zero much faster and the terms containing the logarithm must be omitted.

Consequently, when the terms of Eq. (11) that contain the logarithm are small compared to $\Theta_w - 1$, we can use the following relation to compute the concentrational polarization:

$$\frac{d}{dV\xi}\,[(1-V\xi)\,(\Theta_w-1)] = \frac{\mathrm{Pe}^2V^2}{3}\,[1-(1-\varphi)\,\Theta_w].$$

Having differentiated this relation, we obtain an ordinary linear differential equation of first order, which, together with the boundary condition $\Theta_w = 1$ at $\xi = 0$, gives the solution

$$\Theta_w = \left[1-\varphi-\frac{3}{\mathrm{Pe}^2\,V^2}\right]^{-1}\left(1-\frac{3}{\mathrm{Pe}^2\,V^2}-\varphi\,(1-V\xi)^{(1-\varphi)\frac{\mathrm{Pe}^2V^2}{3}-1}\right). \tag{13}$$

Formula (13) shows the existence of two fundamentally important limiting cases in the case of laminar ultrafiltration. The value of $1-\varphi$ may be small or large compared to the combination $3/\mathrm{Pe}^2V^2$, which characterizes the relationship between the convective and diffusion mass transfer.

We consider the case

$$1-\varphi \ll 3/\mathrm{Pe}^2\,V^2. \tag{14}$$

By virtue of the fact that in the case of ultrafiltration $\mathrm{Pe}V = 10\text{–}10^3$, condition (14) means almost ideal selectivity for the membrane ($\varphi \cong 1$), and then Eq. (13) yields

$$\Theta_w = 1 + \frac{\mathrm{Pe}^2\,V^2V\xi}{3\,(1-V\xi)}, \tag{15}$$

i.e., the concentration of the dissolved substance on the membrane increases continuously with increase in the distance from the entrance of the channel.

Let us return to Eq. (11), assuming that $\varphi = 1$. After integration we obtain

$$\Theta_w - \ln\Theta_w - \frac{1}{2}\,(\ln\Theta_w)^2 = 1 + \frac{\mathrm{Pe}^2\,V^2V\xi}{3\,(1-V\xi)}. \tag{16}$$

When $\Theta_w \gg \ln\Theta_w$, which was assumed in deriving formula (13), it can easily be seen that relation (16) goes over into relation (15).

We now go to the study of the opposite case, where

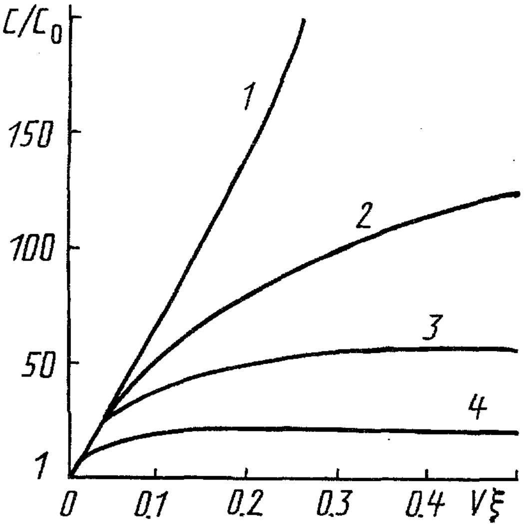

Fig. 1. Distribution of the concentration of an admixture on the wall along the channel length in relation to the selectivity of the membrane for Pe $V = 40$: 1) $\varphi = 1$; 2) 0.99; 3) 0.98; 4) 0.95.

$$1 - \varphi \gg 3/\mathrm{Pe}^2 V^2 . \tag{17}$$

With this condition, as follows from Eq. (13), the concentration of the dissolved substance on the membrane builds up rapidly with an increase in the distance from the entrance of the channel:

$$\Theta_w = \frac{1}{1-\varphi}\left(1 - \varphi\,(1 - V\xi)^{(1-\varphi)\,\mathrm{Pe}^2 V^2/3}\right) \tag{18}$$

and attains the limiting value

$$\Theta_w = \frac{1}{1-\varphi},$$

which remains constant in the remaining part of the slit channel.

We note that formulas (16) and (18) for the limiting cases considered agree very satisfactorily with the numerical solution of the initial system of equations (1)-(4).

Relation (13) also agrees satisfactorily in the region of its applicability with numerical calculations, some results of which are presented in Fig. 1.

In conclusion, we present formulas that establish the relationship between the true selectivity φ and the observed selectivity $\varphi_{\mathrm{ob}} = 1 - C_f/C_0$. When $\Theta_w \simeq 1$, Eq. (12) gives

$$\varphi_{\mathrm{ob}} = \varphi\,[1 - (1-\varphi)\,(2\,\mathrm{Pe}^2 V^2 V\xi)^{1/3}] . \tag{19}$$

The case described by formula (16) corresponds to the condition $\varphi_{\mathrm{ob}} = \varphi = 1$. Finally, from relation (18) we find the relation

$$\varphi_{\mathrm{ob}} = \varphi\,(1 - V\xi)^{(1-\varphi)\,\mathrm{Pe}^2 V^2/3} , \tag{20}$$

which, for a moderate length of the channel, is transformed to the form

$$\varphi_{\mathrm{ob}} = \varphi \exp\,[1 - (1-\varphi)\,\mathrm{Pe}^2 V^2 V\xi/3] . \tag{21}$$

From Eq. (21) it follows that with an increase in the transmembrane velocity the observed selectivity falls to zero following an exponential law.

NOTATION

$\xi = x/h$, $\eta = y/h$, dimensionless longitudinal and transverse coordinates; $u = \hat{u}/u_0$, $v = \hat{v}/u_0$, dimensionless components of the velocity vector; h, half-height of the plane channel; u_0, mean velocity at the channel entrance; $\mathrm{Re} = u_0 h/\nu$, Reynolds number; ν, kinematic viscosity coefficient; $\mathrm{Pe} = u_0 h/D$, diffusional Peclet number; D, diffusion coefficient; $\Theta = C/C_0$, dimensionless concentration of the dissolved substance at the channel inlet; Θ_w, dimensionless concentration of the dissolved substance on the membrane; $V = \hat{V}/u_0$, transmembrane velocity; $\varphi = 1$, ideal selectivity of a membrane; C_f, concentration of the dissolved substance in the filtrate.

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